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Three Body Interactions, Angular Momentum and Black Hole Moduli Spaces

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ABSTRACT

We investigate the dynamics of a pair of (4+1)-dimensional black holes in the moduli approximation and with fixed angular momentum. We find that spinning black holes at small separations are described by the de Alfaro, Fubini and Furlan model. For more than two black holes, we find an explicit expression for the three-body interactions in the moduli metric by associating them with the one-loop three-point amplitude of a four-dimensional ϕ^3 theory. We also investigate the dynamics of a three black hole system in various approximations.

1. Introduction

In the past two years there has been a revival in the interest of understanding the geometry of black hole moduli spaces in the context of supergravity following some earlier work in [1, 2,3]. This was facilitated by some new developments in the the construction of actions for one-dimensional supersymmetric sigma models [4] and the realization that the geometry of the moduli spaces of black holes will involve a connection with torsion [5]. The motivation behind this revival is the hope that investigating these moduli spaces will lead to the understanding of $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence and unravel some novel features in multi-black hole mechanics like bound states, in parallel with similar developments for BPS monopoles. The effective theories of the graviphoton and the Reissner-Nordström multi-black holes were constructed in [6,7] and the effective theories of all static four- and five-dimensional black holes that preserve four supersymmetry charges were constructed in [8, 9]. For small black hole separations, the effective theory of multi-black holes exhibits a $D(2,1;0)$ superconformal symmetry [6,7,9]. Aspects of the multi-black hole quantum mechanics utilizing the $D(2,1;0)$ superconformal symmetry have been investigated in [10, 11].

It has been known for sometime that the (4+1)-dimensional black holes which preserve four supersymmetries have an $AdS_2 \times S^3$ near horizon geometry. A calculation in [12, 13] has revealed that the dynamics of a probe with fixed angular momentum near a black hole with the above near horizon geometry is described by de Alfaro, Fubini and Furlan conformal (DFF) model [14]. The coupling of the conformal model is related to the conserved angular momentum. Using this, it was later argued in [15] that the conformal Calogero model is a candidate for the dual superconformal theory in the context of $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence.

One of the results of this paper is to recover the DFF model from the dynamics of multi-black holes as described by the moduli metric. We shall mostly focus on the dynamics of the supersymmetric black holes of the STU model. For two black holes, we shall show that, if the center of mass motion decouples from the relative motion of the system, then the relative angular momentum of the pair is conserved. Further we shall find that the radial relative motion of the two black hole system is described by a sigma model with a scalar potential which is bell shaped. In particular, we shall show that for non-vanishing relative angular momentum and depending on the energy and original separation of the black hole pair, the two black holes either become dynamically well separated or dynamically approach each other. In addition, we shall find that for small black hole separations the effective theory of a black hole pair with non-vanishing relative angular momentum is given by the DFF model.

We shall also extend the above analysis to systems that involve more than two black holes. For this, we shall give an explicit expression for the black hole moduli metric that involves all the three body interactions. In particular, we shall find that part of the three body interactions can be computed from the one-loop three point amplitude of a four-dimensional ϕ^3-theory. We shall use several approximations to study the dynamics of three black holes. In particular, we shall investigate the system in the limit that two of the black holes are clustered in to a binary while the third one is further away. In this approximation, the effective theory of the system for small black hole separations and with non-vanishing relative angular momentum is also given by a sigma model with a scalar potential which however is not of Calogero type.

This paper is organized as follows: In section two, the moduli metric of a system of STU black holes is given. In section three, the dynamics of a system of two STU black holes is investigated. In section four, the dynamics of a system of more than two black holes is examined.

2. Moduli Metric of (4+1)-dimensional Black holes revisited

It has been found in [8] that the moduli metric of (4+1)-dimensional black holes coupled to any number of vectors which preserve four supersymmetries can be determined from the components of the black hole metric. Suppose that the spacetime metric that describes N black holes located at $\{\mathbf{y}_A \in \mathbb{R}^4; A = 1, \ldots, N\}$ is

$$ds^2 = -A^2(\mathbf{x}, \mathbf{y}_A)dt^2 + B^2(\mathbf{x}, \mathbf{y}_A)ds^2(\mathbb{R}^4) \ , \tag{2.1}$$

where t is the time coordinate and $\mathbf{x} \in \mathbb{R}^4$ are the space coordinates. Then the metric on the moduli space is determined by the moduli potential

$$\mu(\mathbf{y}_A) = \int_{\mathbb{R}^4} d^4x \, A^{-2}B^2(\mathbf{x}, \mathbf{y}_A) \tag{2.2}$$

as

$$ds^2 = \big(\partial_{mA}\partial_{nB} + \sum_{r=1}^{3}(I_r)^k{}_m(I_r)^\ell{}_r\partial_{kA}\partial_{\ell B}\big)\mu \ dy^{mA}dy^{nB} \ , \tag{2.3}$$

where $\{I_r; r = 1, 2, 3\}$ is a constant hypercomplex structure on $\mathbb{R}^4$ associated with a basis of self-dual two-forms and $k, \ell, m, n = 1, \ldots, 4$. The moduli metric can also be written as

$$ds^2 = \big(\delta_{mn}U_{AB} + \sum_{s=1}^{3}(V_s)_{AB}\,(J_s)_{mn}\big)dy^{mA}dy^{nB} \ , \tag{2.4}$$

where $\{J_s; s = 1, 2, 3\}$ is a constant hypercomplex structure on $\mathbb{R}^4$ associated with a basis of anti-self-dual two-forms and

$$\begin{aligned} U_{AB} &= \delta^{mn}\partial_{mA}\partial_{nB}\mu \\ \sum_{s=1}^{3} V_s(J_s)_{mn} &= \big(\partial_{mA}\partial_{nB}\mu - (n, m)\big) - \epsilon_{mn}{}^{k\ell}\partial_{kA}\partial_{\ell B}\mu \ . \end{aligned} \tag{2.5}$$

To show this, one uses uses the identity

$$\sum_{r=1}^{3}(I_r)^{\ell}{}_m(I_r)^k{}_n = \delta_{mn}\delta^{\ell k} - \delta^k{}_m\delta^{\ell}{}_n - \epsilon_{mn}{}^{\ell k} \tag{2.6}$$

and expands the anti-self-dual part of the moduli metric in the J_s basis; the spatial indices are raised and lowered with respect to the Euclidean metric on $\mathbb{R}^4$. As we shall see the components of the moduli metric corresponding to U_{AB} are diagonal in the pair-wise black hole separations while the terms of the moduli metric proportional to $(V_s)_{AB}$ are off-diagonal.

The effective theory of black holes associated with moduli metric (2.3) has $N = 4B$ one-dimensional supersymmetry [4, 5]. The black hole moduli space is a hyper-Kähler manifold with torsion (HKT) [16]. The torsion appears in the couplings of the fermion of the effective theory.

In what follows, we shall be concerned with the dynamics of graviphoton and STU model black holes. Since the graviphoton black holes are a special case of the STU model black holes, we shall describe first the moduli metric of STU black holes. Then we shall give the limit in which the moduli metric of the graviphoton black holes arises from the STU ones. The moduli potential for the black holes of the STU model is

$$\mu = \int_{\mathbb{R}^4} d^4x\, H_1 H_2 H_3 \tag{2.7}$$

where

$$H_i = h_i + \sum_{A=1}^{N} \frac{\lambda_{iA}}{|\mathbf{x} - \mathbf{y}_A|^2} \tag{2.8}$$

for $i = 1, 2, 3$ which are harmonic functions on $\mathbb{R}^4$. The constants $\{h_i; i = 1, 2, 3\}$ are related to the asymptotic values of the two scalars of the theory and the constants $\{\lambda_{iA}; i = 1, 2, 3; A = 1, \dots, N\}$ are interpreted as the charges of the A black-hole with respect to the i-th Maxwell gauge potential; the STU model has three Maxwell fields. For the masses of the black holes to be positive and for the black hole solution not to have naked and other singularities, we take $h_i, \lambda_{iA} > 0$. Using the moduli potential (2.7), we find that the moduli metric of the STU black holes [8] can be rewritten as

$$
\begin{aligned}
ds^2 = {} & V_3 \sum_A [h_2 h_3 \lambda_{1A} + h_1 h_3 \lambda_{2A} + h_1 h_2 \lambda_{3A}] |d\mathbf{y}_A|^2 \\
& + V_3 \sum_{A \neq B} [h_2 \lambda_{1A} \lambda_{3B} + h_1 \lambda_{2A} \lambda_{3B} + h_3 \lambda_{1A} \lambda_{2B}] \frac{|d\mathbf{y}_A - d\mathbf{y}_B|^2}{|\mathbf{y}_A - \mathbf{y}_B|^2} \\
& + \frac{V_3}{4} \sum_{\{A \neq B\}, C} \rho_{ABC} |d\mathbf{y}_A - d\mathbf{y}_B|^2 \Big[\frac{1}{|\mathbf{y}_A - \mathbf{y}_C|^2 |\mathbf{y}_A - \mathbf{y}_B|^2} \\
& \quad + \frac{1}{|\mathbf{y}_B - \mathbf{y}_C|^2 |\mathbf{y}_A - \mathbf{y}_B|^2} - \frac{1}{|\mathbf{y}_A - \mathbf{y}_C|^2 |\mathbf{y}_B - \mathbf{y}_C|^2} \Big] \\
& - \frac{2}{3} \sum_{A \neq B \neq C} \int d^4x \, \rho_{ABC} \frac{\big[(dy_A - dy_C)^{[m} (dy_B - dy_C)^{n]} \big]^-}{|\mathbf{x} - \mathbf{y}_C|^2} \times \\
& \qquad \partial_m \Big(\frac{1}{|\mathbf{x} - \mathbf{y}_A|^2} \Big) \partial_n \Big(\frac{1}{|\mathbf{x} - \mathbf{y}_B|^2} \Big) \ ,
\end{aligned}
\tag{2.9}
$$

where V_3 is the volume of the unit three sphere, $\big[(dy_A - dy_C)^{[m} (dy_B - dy_C)^{n]} \big]^-$ denotes the anti-self-dual projection of $(dy_A - dy_C)^{[m} (dy_B - dy_C)^{n]}$, and

$$
\rho_{ABC} = \big[\lambda_{1A} \lambda_{2B} \lambda_{3C} + \lambda_{1C} \lambda_{2A} \lambda_{3B} + \lambda_{1B} \lambda_{2C} \lambda_{3A} + (A \leftrightarrow B) \big] \ . \tag{2.10}
$$

The moduli metric has a free term for N particles, and two-and three-body velocity dependent interactions. Observe that part of the moduli metric that contains three body interactions is not given explicitly since the last term in (2.9) involves an integration over the spatial coordinates $\mathbf{x}$ which has not been carried out. The three body interactions in the moduli metric that are given explicitly are *diagonal* in the pair-wise black hole separations, $|d\mathbf{x}_A - d\mathbf{y}_B|$, while the term that involves the integral are *off-diagonal*. We remark that if the masses of the black holes $m_A = V_3(h_2 h_3 \lambda_{1A} + h_1 h_3 \lambda_{2A} + h_1 h_2 \lambda_{3A})$ are not equal, then the centre of mass motion does not decouple from the dynamics of the system.

The graviphoton black holes arise in the case where $H_1 = H_2 = H_3 = H$. The presence of one independent harmonic function implies that the two scalars of the STU black hole solutions are constant. Moreover the black holes are charged with respect to a single Maxwell gauge potential which is a linear combination of the three gauge potentials of the STU model. This single gauge potential is identified with the graviphoton of the simple (4+1)-dimensional supergravity. The moduli metric of the graviphoton black holes [6] is given from that of the STU model black holes (2.9) by setting $h_1 = h_2 = h_3 = h$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

At small black hole separations, the dynamics of the system is dominated by the three body interactions. The moduli metric in this limit is given as in (2.9) but with $h_i = 0$ for $i = 1, 2, 3$. The effective theory of supersymmetric black holes apart from the kinetic term involving the moduli metric, it also contains various fermionic terms. In what follows, we shall focus on the classical dynamics governed by the moduli metric only.

3. Two Black Hole System

3.1 Moduli Metric

The moduli metric for two black holes can be written as

$$\begin{aligned} ds^2 = m_1|d\mathbf{y}_1|^2 + m_2|d\mathbf{y}_2|^2 + g_{(2)}\frac{|d\mathbf{y}_1 - d\mathbf{y}_2|^2}{|\mathbf{y}_1 - \mathbf{y}_2|^2} \\ + g_{(3)}\frac{|d\mathbf{y}_1 - d\mathbf{y}_2|^2}{|\mathbf{y}_1 - \mathbf{y}_2|^4} \ , \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} m_1 &= V_3[h_2h_3\lambda_{11} + h_1h_3\lambda_{21} + h_1h_2\lambda_{31}] \\ m_2 &= V_3[h_2h_3\lambda_{12} + h_1h_3\lambda_{22} + h_1h_2\lambda_{32}] \\ g_{(2)} &= V_3[h_2\lambda_{11}\lambda_{32} + h_1\lambda_{21}\lambda_{32} + h_3\lambda_{11}\lambda_{22} + h_2\lambda_{12}\lambda_{31} + h_1\lambda_{22}\lambda_{31} + h_3\lambda_{12}\lambda_{21}] \\ g_{(3)} &= \frac{V_3}{2}(\rho_{122} + \rho_{211}) \ . \end{aligned} \tag{3.2}$$

Using the conditions that we have put on the parameters of the classical solution, we find that $m_1, m_2, g_{(2)}, g_{(3)} > 0$. In particular for the graviphoton black holes, we have

$$\begin{aligned} m &= 3\ V_3\ h^2\ \lambda \\ g_{(2)} &= 6\ V_3\ h\ \lambda^2 \\ g_{(3)} &= 6\ V_3\ \lambda^3\ . \end{aligned} \tag{3.3}$$

The term that contains the integral in (2.9) does not contribute in this case. In the limit of small separations, the moduli metric becomes

$$ds^2 = g_{(3)} \frac{|d\mathbf{y}_1 - d\mathbf{y}_2|^2}{|\mathbf{y}_1 - \mathbf{y}_2|^4}\ . \tag{3.4}$$

A direct observation reveals that if the masses m_1, m_2 of the two black holes are different, then the centre of mass motion of the two black holes does *not* decouple from the relative motion of the system. For the graviphoton black holes the centre of mass motion decouples as well as for those STU black holes for which $\lambda_{i1} = \lambda_{i2}$ for $i = 1, 2, 3$. However for *generic* STU black holes, the centre of mass motion does *not* decouple.

Suppose that $m = m_1 = m_2$ and the center of mass motion decouples from the relative one. In this case we set $\mathbf{r} = \frac{\mathbf{y}_1 - \mathbf{y}_2}{2}$ and $\mathbf{u} = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}$ where $\mathbf{u}$ is the position of the center of mass and $\mathbf{r}$ is the relative position of the two black holes. The moduli metric for such a two black hole system can be rewritten as

$$ds^2 = 2m|d\mathbf{u}|^2 + 2m|d\mathbf{r}|^2 + g_{(2)} \frac{|d\mathbf{r}|^2}{|\mathbf{r}|^2} + 4g_{(3)} \frac{|d\mathbf{r}|^2}{|\mathbf{r}|^4}\ . \tag{3.5}$$

3.2 Dynamics at Fixed Angular Momentum

The moduli metric of the two black hole system is invariant under all $SO(4)$ rotations acting on the positions of the black holes with infinitesimal transformations

$$\delta \mathbf{y}_A = \omega \mathbf{y}_A \ , \tag{3.6}$$

where ω is a 4×4 skew-symmetric matrix. For those black hole systems for which the relative motion decouples for the centre of mass motion, the part of the moduli metric which describes the relative motion is also invariant under the $SO(4)$ rotational symmetry $\delta \mathbf{r} = \omega \mathbf{r}$. This is the case on which we shall focus our attention.

Changing coordinates from Euclidean to angular, we write the relative moduli metric as

$$ds^2 = \phi^2(r)(ds^2 + r^2 ds^2(S^3)) \tag{3.7}$$

where $ds^2(S^3)$ can be parameterized with respect to the right- invariant one forms σ^i, $ds^2(S^3) = \sum_{i=1}^{3}(\sigma^i)^2$, where $d\sigma^i = -\epsilon^i{}_{jk}\sigma^j \wedge \sigma^k$ and

$$\phi^2(r) = 2m + \frac{g_{(2)}}{r^2} + \frac{4g_{(3)}}{r^4} \ . \tag{3.8}$$

Observe that since $m_1, m_2, g_{(2)}, g_{(3)} > 0$, then $\phi^2 > 0$.

To be explicit we write the metric on the relative displacement 3-sphere as

$$ds^2(S^3) = d\theta^2 + \sin^2\theta \big(d\beta^2 + \sin^2\beta d\alpha^2\big) \tag{3.9}$$

where the relative displacement co-ordinates are

$$\mathbf{r} = \begin{pmatrix} r\sin\theta\sin\beta\sin\alpha \\ r\sin\theta\sin\beta\cos\alpha \\ r\sin\theta\cos\beta \\ r\cos\theta \end{pmatrix} \tag{3.10}$$

and the right invariant 1-forms are

$$
\begin{aligned}
\sigma^1 =& -\sin\beta\cos\alpha d\theta + \sin\theta\sin\beta(\sin\alpha\cos\theta + \sin\theta\cos\alpha\cos\beta)d\alpha \\
& -\sin\theta(-\sin\alpha\sin\theta + \cos\beta\cos\alpha\cos\theta)d\beta \\
\sigma^2 =& -\sin\beta\sin\alpha d\theta - \sin\theta\sin\beta(-\sin\theta\cos\beta\sin\alpha + \cos\alpha\cos\theta)d\alpha \\
& +\sin\theta(-\sin\theta\cos\alpha - \cos\theta\cos\beta\sin\alpha)d\beta \\
\sigma^3 =& -\cos\beta d\theta + \sin\beta\sin\theta(\cos\theta d\beta - \sin\theta\sin\beta d\alpha)\ .
\end{aligned}
\tag{3.11}
$$

The conserved angular momentum associated with the $SO(3)$ subgroup of $SO(4)$ that leaves σ^i invariant is

$$
J^i = \phi^2 r^2 \dot{\sigma}^i\ , \tag{3.12}
$$

where now $\dot{\sigma}^i dt$ is the pull-back of σ^i on the worldline and J^i is conserved, $\frac{d}{dt}J^i = 0$. The equation for the radial motion is

$$
-\frac{d}{dt}[\phi^2\frac{d}{dt}r] + \frac{J^2}{r^3\phi^2} + \partial_r\phi^2\big((\frac{d}{dt}r)^2 + \frac{J^2}{r^2\phi^4}\big) = 0\ . \tag{3.13}
$$

The equations of motion associated with the angular coordinates simply imply the conservation of angular momentum. The equation of motion (3.13) is that of a non-relativistic particle with Lagrangian

$$
L = \frac{1}{2}\phi^2(\frac{d}{dt}r)^2 - V(r)\ , \tag{3.14}
$$

where

$$
V(r) = \frac{J^2}{2}\frac{1}{r^2\phi^2(r)}\ . \tag{3.15}
$$

In fact, this potential is associated with the superpotential $W = \log r$ as

$$
V(r) = \frac{J^2}{2}\phi^{-2}(\partial_r W)^2\ , \tag{3.16}
$$

as it may have been expected because the original theory is supersymmetric.

The energy of the system associated with the Lagrangian (3.14) is

$$E = \frac{1}{2}\phi^2(\frac{d}{dt}r)^2 + \frac{J^2}{2}\frac{1}{r^2\phi^2(r)} \tag{3.17}$$

and it is conserved.

There are two cases to consider regarding the dynamics of the system with potential (3.16). First if $J^2 = 0$, then V vanishes. The two black holes can be located at any point in $\mathbb{R}^4$ provided that $E = 0$. The relative dynamics of the system in this case is determined by solving the equation

$$\frac{d}{dt}r = \pm\frac{\sqrt{2E}}{\phi} \ . \tag{3.18}$$

Next suppose that $J^2 \neq 0$. The potential $V(r) > 0$ and $V(r) \to 0$ as $r \to 0$ and $r \to +\infty$. The potential $V(r)$ has two critical points on the positive real line at $r = 0$ and $r_*^2 = (2g_{(3)}/m)^{\frac{1}{2}}$, as can be easily seen by computing $\partial_r V = 0$, ie $V(r)$ has a bell shape. The critical point at $r = r_* > 0$ is a global maximum. The value of the potential at the maximum is $V_* = V(r_*) = \frac{J^2}{8\sqrt{2mg_{(3)}+2g_{(2)}}}$. If the black holes with relative angular momentum $J^2 \neq 0$ are separated by less than the critical distance $r < r_*$ and have energy $E < V_*$, then they roll down the potential towards $r \to 0$ and so their separation is dynamically reduced*. On the other hand if the black holes are separated by $r > r_*$ and have energy $E < V_*$, then they will again roll down the other side of the potential towards $r \to +\infty$. In this case the two black holes become dynamically well separated.

The equation of motion for black holes with $J^2 \neq 0$ is

$$\frac{d}{dt}r = \pm\frac{1}{\phi(r)}\sqrt{2E - \frac{J^2}{r^2\phi^2}} \ . \tag{3.19}$$

In the limit of small black hole separations, the dynamics along the radial direction simplifies considerably. In this case $\phi^2 = 4g_{(3)}/r^4$. Before one proceeds with the

$\star$ We remark that if the initial separation is close to the critical distance then by the time that the black hole have approached $r \to 0$, the non-relativistic approximation may not be valid.

calculation above, it is best to change the radial coordinate as $q = 2\sqrt{g_{(3)}}/r$. In this case, the moduli metric becomes $ds^2 = dq^2 + q^2 ds^2(S^3)$, ie the standard Euclidean metric on $\mathbb{R}^4$ but in angular coordinates.

Focusing on the dynamics of the two black hole system with non-vanishing relative angular momentum, we find that the associated Lagrangian which describes the radial motion is

$$L = \frac{1}{2}[(\frac{d}{dt}q)^2 - \frac{J^2}{q^2}] \ . \tag{3.20}$$

This is precisely the Lagrangian of the DFF model. Therefore we have shown that the probe computation of [12, 13] coincides with the result obtained from the moduli approach.

4. Black Hole Three Body Interactions

4.1 Three body interactions and ϕ^3 theory

To make progress towards an explicit expression for the moduli metric of $N > 2$ (4+1)-dimensional black holes, we have to evaluate the integral in (2.9) which involves the off-diagonal three body interactions. This contribution to the moduli metric can be rewritten as

$$ds^2_{OD} = -\frac{2}{3} \sum_{A \neq B \neq C} \rho_{ABC} \left[(dy_A - dy_C)^{[m} (dy_B - dy_C)^{n]}\right]^- \partial_{mA} \partial_{nB} \mathcal{A}(\mathbf{y}_A, \mathbf{y}_B, \mathbf{y}_C) \ , \tag{4.1}$$

where

$$\mathcal{A}(\mathbf{y}_A, \mathbf{y}_B, \mathbf{y}_C) = \int d^4x \, \frac{1}{|\mathbf{x} - \mathbf{y}_C|^2} \frac{1}{|\mathbf{x} - \mathbf{y}_A|^2} \frac{1}{|\mathbf{x} - \mathbf{y}_B|^2} \ . \tag{4.2}$$

Observe that since the positions $\mathbf{y}_A$, $\mathbf{y}_B$, $\mathbf{y}_C$ are all distinct, $\mathcal{A}$ is finite. It can be immediately recognized that $\mathcal{A}$ is the one-loop three point green function of a massless ϕ^3 theory. For this, $\mathbf{x}$ is identified as the loop momentum $\mathbf{k}$ and $\mathbf{p}_{AB} = \mathbf{y}_A - \mathbf{y}_B$ are identified as the three incoming momenta. To compute this integral, we shall use field theory techniques and terminology. After changing variables, $\mathcal{A}$ can be written as

$$\mathcal{A}(\mathbf{p}_{AC}, \mathbf{p}_{BC}) = \int d^4k \, \frac{1}{|\mathbf{k}|^2} \frac{1}{|\mathbf{k} - \mathbf{p}_{AC}|^2} \frac{1}{|\mathbf{k} - \mathbf{p}_{BC}|^2} \ . \tag{4.3}$$

Applying standard field theory methods, $\mathcal{A}$ can be expressed as

$$\mathcal{A} = 2 \int d^4k \int_0^1 d\alpha \int_0^\alpha d\beta [|\mathbf{k}-\mathbf{p}_{BC}|^2\beta + (\alpha-\beta)|\mathbf{k}-\mathbf{p}_{AC}|^2 + |\mathbf{k}|^2(1-\alpha)]^{-3} \ . \tag{4.4}$$

After again changing variables $\mathbf{k} \to \mathbf{k} - \mathbf{p}_{BC}\beta - \mathbf{p}_{AC}(\alpha - \beta)$ and performing the integration over $\mathbf{k}$, we find that

$$\mathcal{A} = \frac{V_3}{2} \int_0^1 d\alpha \int_0^\alpha d\beta [|\mathbf{p}_{AB}|^2(\alpha-\beta)\beta + (1-\alpha)(\beta|\mathbf{p}_{BC}|^2 + (\alpha-\beta)|\mathbf{p}_{AC}|^2)]^{-1} \ . \tag{4.5}$$

It is next convenient to change co-ordinates by defining $\beta = \alpha w$, and to set $m = 0$. Then the integral may be rewritten as

$$\mathcal{A} = \frac{V_3}{2} \int_0^1 \int_0^1 d\alpha dw \frac{1}{|\mathbf{p}_{AB}|^2\alpha w(1-w) + (1-\alpha)(|\mathbf{p}_{BC}|^2 w + (1-w)|\mathbf{p}_{AC}|^2)} \ . \tag{4.6}$$

Carrying out the α-integral, we obtain

$$\mathcal{A} = -\frac{V_3}{2} \int_0^1 dw \frac{\log\left(\frac{|\mathbf{p}_{AB}|^2 w(1-w)}{|\mathbf{p}_{BC}|^2 w + |\mathbf{p}_{AC}|^2(1-w)}\right)}{\left[|\mathbf{p}_{AB}|^2 w^2 + (|\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2 - |\mathbf{p}_{AB}|^2)w + |\mathbf{p}_{AC}|^2\right]} \ . \tag{4.7}$$

The full expression for this integrand is given in the appendix. As it is difficult to obtain an insight of the black hole dynamics from the resulting rather complicated expression, we shall use an approximation to investigate some of its properties.

4.2 Black Hole Binaries

In the black hole binary approximation, black holes A and B are close together while the black hole C is further way. In such an approximation, $|\mathbf{p}_{AB}|^2 \ll |\mathbf{p}_{AC}|^2$ and $|\mathbf{p}_{AB}|^2 \ll |\mathbf{p}_{BC}|^2$. This implies that $||\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2| \ll |\mathbf{p}_{AC}|^2$ and also $||\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2| \ll |\mathbf{p}_{BC}|^2$. It is also useful to change co-ordinates again by setting $w = \frac{1}{2} + v$. Then the symmetry of $\mathcal{A}$ under the interchange $A \leftrightarrow B$ is made manifest as

$$\mathcal{A} = -\frac{V_3}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dv \frac{\log\left(\frac{|\mathbf{p}_{AB}|^2(\frac{1}{4}-v^2)}{v(|\mathbf{p}_{BC}|^2-|\mathbf{p}_{AC}|^2)+\frac{1}{2}(|\mathbf{p}_{BC}|^2+|\mathbf{p}_{AC}|^2)}\right)}{(v^2-\frac{1}{4})|\mathbf{p}_{AB}|^2 + v(|\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2) + \frac{1}{2}(|\mathbf{p}_{AC}|^2 + |\mathbf{p}_{BC}|^2)} \ . \tag{4.8}$$

It is clear that under the above assumptions $\frac{|\mathbf{p}_{AB}|^2}{|\mathbf{p}_{AC}|^2+|\mathbf{p}_{BC}|^2}$ is a small dimensionless parameter which we can use to expand the integral out. In particular we find that

$$\begin{aligned}\mathcal{A} = & -\frac{V_3}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dv \frac{2}{|\mathbf{p}_{AC}|^2 + |\mathbf{p}_{BC}|^2}\times \\ & \Big[\big[1 - \frac{2}{|\mathbf{p}_{AC}|^2 + |\mathbf{p}_{BC}|^2}\big((v^2 - \frac{1}{4})|\mathbf{p}_{AB}|^2 + v(|\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2)\big)\big]\times \\ & \big[\log\big(\frac{2|\mathbf{p}_{AB}|^2(\frac{1}{4}-v^2)}{|\mathbf{p}_{BC}|^2 + |\mathbf{p}_{AC}|^2}\big) - \frac{2v(|\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2)}{|\mathbf{p}_{AC}|^2 + |\mathbf{p}_{BC}|^2}\big] + O\big(\big(\frac{|\mathbf{p}_{AB}|^2}{|\mathbf{p}_{AC}|^2 + |\mathbf{p}_{BC}|^2}\big)^2\big)\Big] \ .\end{aligned} \tag{4.9}$$

The leading order term in this expression is given by

$$\mathcal{A} = -V_3 \frac{\log\left(\frac{|\mathbf{p}_{AB}|^2}{|\mathbf{p}_{BC}|^2+|\mathbf{p}_{AC}|^2}\right)}{|\mathbf{p}_{BC}|^2 + |\mathbf{p}_{AC}|^2} \ . \tag{4.10}$$

Acting on this term with the differential operator $\partial_{mA}\partial_{nB} - \partial_{nA}\partial_{mB}$, we obtain from this the leading order off-diagonal contribution to the relevant component of the metric which is given by

$$ds^2_{OD} \sim -\frac{8V_3}{3}\rho_{ABC}\frac{(y_A-y_B)_m(y_A+y_B-2y_C)_n}{|\mathbf{p}_{AB}|^2(|\mathbf{p}_{AC}|^2+|\mathbf{p}_{BC}|^2)^2}\left[(dy_A-dy_C)^{[m}(dy_B-dy_C)^{n]}\right]^- \ , \tag{4.11}$$

where $\sim$ indicates that the moduli metric is corrected by higher order terms. We observe that this expression is of order $\frac{1}{|\mathbf{p}_{AB}||\mathbf{p}_{AC}|^3}$. The remaining 3-body interactions

$$\begin{aligned} ds^2_D =&\frac{V_3}{4}\sum_{\{D\neq E\},F}\rho_{DEF}|d\mathbf{y}_D-d\mathbf{y}_E|^2\Big[\frac{1}{|\mathbf{y}_D-\mathbf{y}_F|^2|\mathbf{y}_D-\mathbf{y}_E|^2} \\ &+\frac{1}{|\mathbf{y}_E-\mathbf{y}_F|^2|\mathbf{y}_D-\mathbf{y}_E|^2}-\frac{1}{|\mathbf{y}_D-\mathbf{y}_F|^2|\mathbf{y}_E-\mathbf{y}_F|^2}\Big] \ , \end{aligned} \tag{4.12}$$

which are diagonal in the separations, contain terms of order $\frac{1}{|\mathbf{p}_{AB}|^4}$, $\frac{1}{|\mathbf{p}_{AB}|^2|\mathbf{p}_{AC}|^2}$ and $\frac{1}{|\mathbf{p}_{AC}|^4}$. So, comparing the components of ds^2_{OD} and ds^2_D, it is consistent to consider two different approximations for the moduli metric. First, we can keep the components of ds^2_D which are of order $\frac{1}{|\mathbf{p}_{AB}|^4}$ and $\frac{1}{|\mathbf{p}_{AB}|^2|\mathbf{p}_{AC}|^2}$, and the leading order term in ds^2_{OD}. Alternatively, we may retain only the components of ds^2_D which are of order $\frac{1}{|\mathbf{p}_{AB}|^4}$ and $\frac{1}{|\mathbf{p}_{AB}|^2|\mathbf{p}_{AC}|^2}$.

Clearly the approximation that we are considering applies for a system of three-black holes for which two of them are clustered together to form a binary system while the third is further way. For more than three black holes, the system is more complicated, as three or more black holes can cluster together and so the associated full three-body interaction becomes relevant. However one can envisage the possibility where the black holes cluster in binaries which have separations larger than those of the black holes within the binaries. In such a case the above approximation applies for a system of more than three black holes.

4.3 Angular Momentum and Black Hole Binaries

We can now investigate whether in the black hole binary approximation it is possible to carry out an analysis similar to that which we have performed in section three for the two black hole system. One of the difficulties which emerges is that, although the angular momentum of the whole system is conserved, the angular momentum of each black hole or black hole pair is not. To simplify the analysis, we shall consider the case of three black holes. First, we shall use the approximation of the previous section in which only the components in the moduli metric diagonal in the black hole separations contribute.

Now suppose that the black holes $A = 1$ and $B = 2$ are close together while the black hole $C = 3$ is further away. So if we set $\mathbf{v} = \mathbf{y}_1 - \mathbf{y}_2$ and $\mathbf{w} = \mathbf{y}_1 - \mathbf{y}_3$, we have that $|\mathbf{v}| << |\mathbf{w}|$ and $\mathbf{w} \sim \mathbf{y}_2 - \mathbf{y}_3$. Working in the limit in which the three-body interactions dominate and off-diagonal components in the separations are neglected, we find that the moduli metric is

$$ds^2 \sim f^2(|\mathbf{v}|, |\mathbf{w}|)|d\mathbf{v}|^2 \tag{4.13}$$

where

$$f^2(|\mathbf{v}|, |\mathbf{w}|) = \frac{V_3}{2}\Big(\frac{\rho_{122} + \rho_{211}}{|\mathbf{v}|^4} + \frac{2\rho_{123}}{|\mathbf{v}|^2|\mathbf{w}|^2}\Big) . \tag{4.14}$$

In this approximation the angular momentum of the pair of the $A = 1$ and $B = 2$ black holes is conserved. So we can now use the analysis we have done in section three to investigate the effective potential associated with the system for fixed angular momentum. We find that the associated potential is

$$V(q_1, q_2) = \frac{J^2}{V_3\big[(\rho_{122} + \rho_{211})|q_1|^2 + 2\rho_{123}|q_2|^2\big]} \tag{4.15}$$

where $q_1 = |\mathbf{v}|^{-1}$ and $q_2 = |\mathbf{w}|^{-1}$. Observe that this potential is not of the conformal Calogero type.

Next consider the approximation where the leading term in ds^2_{OD} contributes as well. In such a case the moduli metric for the three black hole system is

$$ds^2 \sim f^2(|\mathbf{v}|, |\mathbf{w}|)|d\mathbf{v}|^2 - \frac{4\rho_{123}}{3|\mathbf{v}|^2|\mathbf{w}|^4}(\mathbf{v} \wedge \mathbf{w}) \cdot (d\mathbf{v} \wedge d\mathbf{w})^- \ , \tag{4.16}$$

using a self-explanatory notation. It is clear from this that the addition of the leading order term in ds^2_{OD} violates the conservation of the angular momentum of each pair of black holes in the system. To conclude, we remark that it is curious that the four-dimensional theory with cubic interactions enters in the investigation of the moduli metric for (4+1)-dimensional black holes. It is not clear whether this is simply a technical coincidence or if there is a more fundamental reason for it.

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APPENDIX : The three-body interaction moduli potential

Here we shall give the expression for the amplitude $\mathcal{A}$. For this we take (4.8) and make the change of variables $v = \frac{z}{2}$. It is also convenient to define $\gamma = \frac{1}{2}|\mathbf{p}_{AB}|^2$, $\tau = |\mathbf{p}_{BC}|^2 - |\mathbf{p}_{AC}|^2$, $\sigma = |\mathbf{p}_{BC}|^2 + |\mathbf{p}_{AC}|^2$. Then

$$\mathcal{A} = -\frac{V_3}{2}\int_{-1}^{1} dz \frac{1}{\gamma z^2 + \tau z + \sigma - \gamma} \log\left(\frac{\gamma(1-z^2)}{\tau z + \sigma}\right)$$

and from the definitions of γ, τ and σ it follows that $|\tau| < 2\gamma$, $2\gamma < \sigma$ and $\sigma > |\tau|$. From these inequalities we observe that $\gamma z^2 + \tau z + \sigma - \gamma > 0$ for $z \in \mathbb{R}$. Defining

$a^\pm = -\frac{\tau}{2\gamma} \pm i\sqrt{\frac{\sigma}{\gamma} - \frac{\tau^2}{4\gamma^2} - 1}$, we have $\gamma z^2 + \tau z + \sigma - \gamma = \gamma(z-a^+)(z-a^-)$. Hence

$$\mathcal{A} = -\frac{V_3}{2\gamma(a^+ - a^-)} \int_{-1}^{1} dz \Big(\frac{1}{z-a^+} - \frac{1}{z-a^-}\Big) \log\Big(\frac{\gamma(1-z^2)}{\tau z + \sigma}\Big) \ .$$

Evaluating this expression we obtain

$$\begin{aligned}
\mathcal{A} = &- \frac{V_3}{2\gamma(a^+ - a^-)} \times \\
&\Big[- \mathrm{dilog}\Big(\frac{2}{1-a^+}\Big) - \mathrm{dilog}\Big(\frac{2}{1+a^+}\Big) - \mathrm{dilog}\Big(\frac{\frac{\sigma}{|\tau|}+1}{a^+ + \frac{\sigma}{|\tau|}}\Big) + \mathrm{dilog}\Big(\frac{\frac{\sigma}{|\tau|}-1}{a^+ + \frac{\sigma}{|\tau|}}\Big) \\
&+ \log\frac{\gamma}{|\tau|}\big(\log(a^+ + 1) - \log(a^+ - 1)\big) + \log\Big(\frac{a^+ + 1}{a^+ - 1}\Big)\big(\log(a^+ + 1) - \log(1 - a^+)\big) \\
&+ \log\Big(a^+ + \frac{\sigma}{|\tau|}\Big)\Big(\log\Big(\frac{a^+ + 1}{a^+ + \frac{\sigma}{|\tau|}}\Big) - \log\Big(\frac{a^+ - 1}{a^+ + \frac{\sigma}{|\tau|}}\Big)\Big) - (a^+ \leftrightarrow a^-)\Big] \ ,
\end{aligned}$$

where dilog denotes the principal branch of the dilogarithm function defined on $\mathbb{C}$ cut along $\Re\,(x) = (-\infty, 0)$, and

$$\mathrm{dilog}\ x = \int_{1}^{x} dt\ \frac{\log t}{1-t} \ .$$

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